Symmetric polynomials

One of our first encounter with symmetric polynomials was through the following problem (Thursday, September 28, 2023):

Problem 1. *Let $p(x) = x^5 + x$ and $q(x) = x^5 + x^2$. Find all $(w, z) \in \mathbf{C}^2$ such that $w \neq z$ and*

$$\begin{cases} p(w) = p(z) \\ q(w) = q(z). \end{cases} \tag{1}$$

Partial proof. It is fairly easy to find a relation between w and z: because $w \neq z$ is supposed, we are really tempted to divide $p(w) - p(z)$ and $q(w) - q(z)$ by $w - z$:

$$P(w, z) = \frac{p(w) - p(z)}{w - z} = \frac{w^5 - z^5 + w - z}{w - z} = \frac{w^5 - z^5}{w - z} + 1 \tag{2}$$

$$Q(w, z) = \frac{q(w) - q(z)}{w - z} = \frac{w^5 - z^5 + w^2 - z^2}{w - z} = \frac{w^5 - z^5}{w - z} + (w + z). \tag{3}$$

Thus $w + z = 1$. Brutal calculations may conclude but there is a smarter and classic way. The sum of the two "roots" (w and z) is known and there is a special situation where the knowledge of the sum and the product of roots determines completely a polynomial : the symmetric polynomials (in two variables, in this case). Hence, knowing the value of wz would allow us to find all the possible values (w, z) such that is it solution of 1, as desired. □

The aim of this paper is not to give a detailed proof of problem 1 (see IMC 2000). We will only survey some of the main properties of symmetric polynomials (using Gourdon's *Algebra*).

Let $\mathbf{A}$ be an unitary commutative ring.

Definition 2. *A polynomial $P \in \mathbf{A}[X_1, \dots, X_n]$ is symmetric if :*

$$P(X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(n)}) = P(X_1, X_2, \dots, X_n) \tag{4}$$

for all $\sigma \in \mathfrak{S}_n$ (σ is a permutation of n elements).

Let's see some examples. We will use SageMath (a free open-source mathematics software system) to check properties and play with symmetric polynomials. First, we have to instantiate the ring on which we're working on. Let's say it is $\mathbf{R}[X, Y, Z]$:

```
R.<X, Y, Z> = PolynomialRing(RR)
```

One of the simplest example (putting aside the null function) is :

```
f = X + Y + Z
f.is_symmetric() # True
```

We can also build more complicated example :

```
f = X^4 + Y^4 + Z^4 + X^2*Y*Z + X*Y^2*Z + X*Y*Z^2
f.is_symmetric() # True
```

We can build incredibly complicated example but it'll only be an illusion because each symmetric polynomial can be expressed as a combination of elementary symmetric polynomials. Understanding this class of symmetric polynomials will be sufficient to deal with general symmetric polynomials.

Definition 3. *The k-th elementary symmetric polynomial $e_k \in \mathbf{A}[X_1, \ldots, X_n]$ is :*

$$e_k(X_1, X_2, \ldots, X_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} X_{i_1} X_{i_2} \cdots X_{i_k}. \tag{5}$$

We will produce the list of the k-th symmetric polynomial for small values of n.

```
R = PolynomialRing(RR, n, 'X')
for i in range(n + 1):
    f = e([i])
    g = f.expand(n, alphabet=['X_' + str(i) for i in range(1, n + 1)])
    print(g)
```

We thus obtain a beautiful Christmas tree (or a space invaders ship) !

n	e_k (with $k \in [\![1, n]\!]$)
1	1 X_1
2	1 $X_1 + X_2$ X_1X_2
3	1 $X_1 + X_2 + X_3$ $X_1X_2 + X_1X_3 + X_2X_3$ $X_1X_2X_3$
4	1 $X_1 + X_2 + X_3 + X_4$ $X_1X_2 + X_1X_3 + X_2X_3 + X_1X_4 + X_2X_4 + X_3X_4$ $X_1X_2X_3 + X_1X_2X_4 + X_1X_3X_4 + X_2X_3X_4$ $X_1X_2X_3X_4$
5	$X_1 + X_2 + X_3 + X_4 + X_5$ $X_1X_2 + X_1X_3 + X_2X_3 + X_1X_4 + X_2X_4 + X_3X_4 + X_1X_5 + X_2X_5 + X_3X_5 + X_4X_5$ $X_1X_2X_3 + X_1X_2X_4 + X_1X_3X_4 + X_2X_3X_4 + X_1X_2X_5 + X_1X_3X_5 + X_2X_3X_5 + X_1X_4X_5 + X_2X_4X_5 + X_3X_4X_5$ $X_1X_2X_3X_4 + X_1X_2X_3X_5 + X_1X_2X_4X_5 + X_1X_3X_4X_5 + X_2X_3X_4X_5$ $X_1X_2X_3X_4X_5$

Our problem is now a *symmetrization problem* : given a symmetric polynomial expressed in the usual basis we want to explicit the same polynomial in the basis of elementary symmetric polynomials. That can be done, for instance, using the "SymmetricReduction[...]" function from WolframAlpha. An implementation has been made for SageMath by Federico Lebrón (the source has been slightly modified to be up-to-date) :

```
def symmetrize(p):
    if not p.is_symmetric(): raise Error(str(p) + " is not a symmetric polynomial.")
    vars = p.variables()
    nvars = len(vars)

    S = SymmetricFunctions(RR)
    e = S.elementary()
    sigmas = [e([i]).expand(nvars, alphabet=vars) for i in range(1, nvars+1)]

    R = PolynomialRing(p.base_ring(), ['sigma_%s' % i for i in range(1, nvars+1)])
```

```
    sigma_vars = R.gens()

    def sym(f):
        if f == 0: return 0
        c = f.lc()
        degrees = f.lm().degrees()

        exps = [degrees[i]-degrees[i+1] for i in range(len(degrees)-1)]
        exps.extend(degrees[-1:])

        g = prod([sigma_vars[i]**exps[i] for i in range(len(exps))])
        gp = prod([sigmas[i]**exps[i] for i in range(len(exps))])

        return c*g + sym(f - c*gp)
    return sym(p)
```

We look at the $n = 2$ case, for convenience we let $\sigma_1 = X + Y$ and $\sigma_2 = XY$.
For instance, suppose $f(X, Y) = X^2 + Y^2 + 6XY - 3X^2Y - 3XY^2$. Then, the symmetrized version of f is $\sigma_1^2 - 3\sigma_1\sigma_2 + 4\sigma_2$. To gain in intuition, at least on $\mathbf{R}[X, Y]$, one may consider constructing many examples.

$P \in \mathbf{R}[X,Y]$	Symmetrized version	$P \in \mathbf{R}[X,Y]$	Symmetrized version
$(X+Y) - X - Y$	0	$X + Y$	σ_1
$(X+Y)^2 - X^2 - Y^2$	$2\sigma_2$	$X^2 + Y^2$	$\sigma_1^2 - 2\sigma_2$
$(X+Y)^3 - X^3 - Y^3$	$3\sigma_1\sigma_2$	$X^3 + Y^3$	$\sigma_1^3 - 3\sigma_1\sigma_2$
$(X+Y)^4 - X^4 - Y^4$	$4\sigma_1^2\sigma_2 - 2\sigma_2^2$	$X^4 + Y^4$	$\sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2$
$(X+Y)^5 - X^5 - Y^5$	$5\sigma_1^3\sigma_2 - 5\sigma_1\sigma_2^2$	$X^5 + Y^5$	$\sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2$
$-X - Y + 8X + 8Y$	$7\sigma_1$	$X^2Y + Y^2X$	$\sigma_1\sigma_2$
$-X^2 - Y^2 + 8X^2Y + 8Y^2X$	$-\sigma_1^2 + 8\sigma_1\sigma_2 + 2\sigma_2$	$X^4Y^2 + Y^4X^2$	$\sigma_1^2\sigma_2^2 - 2\sigma_2^3$
$-X^3 - Y^3 + 8X^3Y^2 + 8Y^3X^2$	$-\sigma_1^3 + 8\sigma_1\sigma_2^2 + 3\sigma_1\sigma_2$	$X^6Y^3 + Y^6X^3$	$\sigma_1^3\sigma_2^3 - 3\sigma_1\sigma_2^4$
$-X^4 - Y^4 + 8X^4Y^3 + 8Y^4X^3$	$-\sigma_1^4 + 8\sigma_1\sigma_2^3 + 4\sigma_1^2\sigma_2 - 2\sigma_2^2$	$X^8Y^4 + Y^8X^4$	$\sigma_1^4\sigma_2^4 - 4\sigma_1^2\sigma_2^5 + 2\sigma_2^6$
$-X^5 - Y^5 + 8X^5Y^4 + 8Y^5X^4$	$-\sigma_1^5 + 8\sigma_1\sigma_2^4 + 5\sigma_1^3\sigma_2 - 5\sigma_1\sigma_2^2$	$X^{10}Y^5 + Y^{10}X^5$	$\sigma_1^5\sigma_2^5 - 5\sigma_1^3\sigma_2^6 + 5\sigma_1\sigma_2^7$

More generally, one can proceed the same way for each symmetric polynomial in n variables.

Theorem 4. *Every symmetric polynomial of $\mathbf{A}[X_1, \ldots, X_n]$ is uniquely determined by a polynomial expression of elementary symmetric polynomials.*

Proof. See Algebra, class n°33, MIT or Charles Walter's course. □

Let's see an usual application of the theorem 4. Set n to 1 and consider the root-expression of a n-th degree polynomial :

$$P_n(X) = \prod_{i=1}^{n}(X - z_i). \tag{6}$$

If we expand it for small values of n, we find :

$$P_1(X) = X - z_1 \tag{7}$$

$$P_2(X) = X^2 - (z_1 + z_2)X + z_1 z_2 \tag{8}$$

$$P_3(X) = X^3 - (z_1 + z_2 + z_3)X^2 + (z_1 z_2 + z_2 z_3 + z_3 z_1)X - z_1 z_2 z_3. \tag{9}$$

We obviously recognize some of the elementary symmetric polynomials e_k ! More generally, one can prove the following result :

$$P(\sigma_1, \sigma_2, \ldots, \sigma_n) := P_n(X) = X^n - \sigma_1 X^{n-1} + \sigma_2 X^{n-2} - \ldots + (-1)^n \sigma_n. \tag{10}$$

We thus have a *duality result* :

$$\begin{array}{ccc} \text{Symmetric polynomial} & & \text{One-variable polynomial} \\ \text{in terms of roots of } P & \longleftrightarrow & \text{in terms of coeffs of } P \\ P(\sigma_1, \sigma_2, \ldots, \sigma_n) & & P_n(X) \end{array}$$

As an other example of this duality, one can the consider generating function of the elementary symmetric polynomials:

$$g(t) := \sum_{k \geq 0} e_k(X_1, X_2, \ldots, X_n) t^k = \prod_{i=1}^{n} (1 + t z_i). \tag{11}$$

We then have:

$$P_n(t) = t^n g(-1/t). \tag{12}$$

We can also attach some special symmetric polynomial functions to a given, for instance, one-variable polynomial. A canonical example is the discriminant. Just think of the well-known quantity $\Delta = b^2 - 4ac$. But $\Delta(a, b, c)$ is not a symmetric polynomial of $\mathbf{A}[a, b, c]$. Instead of the (a, b, c)-triplet we shall consider the roots of z_1 and z_2 of a two-degree polynomial. Computing $(z_1 - z_2)^2$ gives the desired result $b^2 - 4ac$ (up to a division by a, a minor detail without more significance than a scaling constant). One sees that $(z_1 - z_2)^2$ is a symmetric polynomial!

More generally, one defines the discriminant $\Delta(P)$ of $P(X) = \prod_{i=1}^{n} (X - z_i)$ such that :

$$\Delta(P) = \prod_{i<j} (z_i - z_j)^2. \tag{13}$$